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Stochastic Approximation Result for Random Split Variational Inequality Problems in Hilbert Spaces

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Abstract

This study aims at extending the idea of common solutions to problems in classical functional analysis to accommodate situations where there are randomness in the system, as real life problems are, mostly, of this nature. A common solution to random split feasibility and random variational inequality problems, called random split variational inequality problem, is sought through fixed point theory, using a nonexpansive operator. A random type of the two-step Wang's algorithm is used to obtain a unique solution to the problem; and a strong convergence to this unique solution is proven. The result is applied to optimal tax policy problem and is seen to be adequate in solving the problem, yielding tax rates of 14.79% and 13.91% for the two categories of businesses. This result extends, and unifies some established results in the literature on deterministic functional analysis.

Keywords: Random, Split Feasibility, Variational Inequality, Monotone, Hilbert Spaces, Lipschitzian.

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1. Introduction

Intensive researches have been carried out in solving variational inequality problem (VIP), which happens to be one of the core problems in Functional Analysis. This is because it has proven effective in solving problems in many areas, including mechanics, nonlinear programming, game theory and equilibrium problems. It involves finding a point in a closed convex subset of a space, such that certain inequality constraint is satisfied. In the classical case, the VIP is formulated as;

Find $x^* \in C$ such that $\langle Fx^*, x^* - x\rangle \leq 0; \forall\, x \in C$ (1.1)

where C is a closed convex subset of a real Hilbert space; and F is a continuous function, and x^* is the unique solution (Lohawech, Kaewcharoen, & Farajzadeh, 2021). Also see (Ceng, Ansari, & Yao, 2008; Ceng, Teboulle, & Yao, 2010; Fukushima, 1986; Kinderlehrer & Stampacchia, 2000; Yang & Bell, 1997), and the references therein.

The Split Feasibility Problem (SFP), which is applied in inverse problems of intensity-modulated radiation therapy, image reconstruction and signal processing is another important area in Functional Analysis. It involves finding a point in a convex subset of a space, such that under a suitable trasformation, yields a corresponding image in another convex subset.

The SFP, as introduced by (Censor & Elfving, 1994), is formulated as;

Find $x^* \in C\!: Ax^* \in Q$ (1.2)

where $C \in \mathfrak{H}_1$ and $Q \in \mathfrak{H}_2$ are closed convex subsets of the Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively; A is a linear bounded self-adjoining mpping and x^* is the unique solution. For some of the notable works in the area, see (Censor, Elfving, Kopf, & Brtfield, 2005; Byrne, 2002; Censor & Elfving, 1994).

In a bid to solve the VIP and SFP simultaneously, Jung (2016) developed a problem, referred to in literature as Split Variational Inequality Problem, whose solution satisfies both (1.1) and (1.2). The problem is presented as;

Find $x^* \in \Gamma$ such that $\langle Fx^*, x^* - x\rangle \leq 0; \forall\, x \in C$ (1.3)

where Γ is the feasible set of the SFP in (1.2).

As it is the case with many problems in Functional Analysis, the solution to (1.3) can not be found analytically. The solution is, therefore sought for, numerically, using iterative algorithms.

Yamada (2001) developed an algorithm he called the hybrid steepest descent method, which he used to solve a SFP. The algorithm is given as;

$$x_{n+1} = (I - t\mu F)Tx_n \tag{1.4}$$

where $t \in (0,1)$, $\mu \in (0,1)$, F is a continuous function, and T is a nonexpansive mapping. This was developed as a steepest descent type algorithm minimizing certain convex functions. The major feature of the hybrid steepest descent method is that it does not require the closed form expression for the metric projection onto the convex subset, C, denoted by P_C; instead, it requires a closed form expression of a nonexpansive mapping T, whose fixed point set is C.

Wang (2007), motivated by the work of Yamada (2001), developed a new explicit iterative scheme to approximate the fixed point of a nonexpansive mapping T in a Hilbert space. This algorithm has the hybrid steepest descent method, as its fundamental component. The iteration was developed as

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(I - t\mu F)Tx_n, n \geq 0 \tag{1.5}$$

satisfying the conditions;

i. $\alpha \leq \alpha_n \leq \beta$ for some $\alpha, \beta \in (0,1)$
ii. $\mu \in (0,1)$
iii. $\sum_{n=1}^{\infty} \mu_n < \infty$

When the functions are random, as is mostly the case in real life situations, the determinstic methods of solutions become inadequate. In such situations, it is necessary to modify the problem to accommodate the perturbations in the system. Stochastic methods are then employed, where the underlying functions are the expected value of stochastic functions. This makes the random version of (1.3) to be formulated as

Find $x^*(\omega) \in \Gamma$ such that $\langle E[Gx^*(\omega)], x^*(\omega) - x(\omega)\rangle \leq 0; \forall\, x(\omega) \in C$ (1.6)

where $x^*(\omega)$ is a random point in C, ω is the random component of the variable x, G is an integrable function on some measurable space, and E is the integral or expected value of $G[x^*(\omega)]$, with respect to $\rho(\omega)$, a function of the random variable ω. This is the random split variational inequality problem (RSVIP).

Observe that when G is purely deterministic (devoid of randomness), (1.6) can be regarded as (1.3). Even so, G is not always fully observable and the knowledge of G is always scarce for reasons stated in (Udom & Nweke, 2021), (Udom, 2018), which are; (i) the probability distribution of $\rho(\omega)$ may be known but computation of the integral in (1.6) may involve multi-dimensional integration which is computationally difficult, if not impossible; (ii) the random function G is known but the distribution of $\rho(\omega)$ is not known, so that the information on $\rho(\omega)$ can only be obtained using historical data or by sampling; (iii) $E[Gx(\omega)]$ is not observable and it must be approximately evaluated through simulation procedure.

In this work, a solution to the random split variational inequality problem, as developed in (1.6) will be sought, through random fixed point theory. A random version of the two-step Wang's algorithm, used in (Miao & Li, 2008), will be used to obtain strong convergence to the unique solution of (1.6). This will make the result more useful, as most real life problems are random in nature; and will also generalise some of the existing results in literature

2. Definitions and Preliminaries

Definition 2.1 (Hilbert space)

A Hilbert space is a complete inner product space (complete in the metric defined by the inner product $\langle .,. \rangle$). This is the space that the woek is carried out in.

Definition 2.2 (Udom & Nweke, 2021)

Let (Ω, Φ, P) be a complete probability measure space, C be a nonempty subset of a separable Hilbert space, $\mathfrak{H}$ and $T: \Omega \times \mathfrak{H} \to \mathfrak{H}$ is a random operator. The random point $z(\omega): \Omega \to \mathfrak{H}$ is called a fixed point of T if $T(z(\omega), \omega) = z(\omega)$. The set of all fixed points of T is denoted by $Fix(T)$. This is the method of solution that is sought for, in this paper.

Definition 2.3 (Udom, 2018)

Let (Ω, Φ, P) be a complete probability measure space, C be a nonempty subset of a separable Hilbert space, $\mathfrak{H}$. The random operator, $T: \Omega \times \mathfrak{H} \to \mathfrak{H}$, is;

i. κ-Lipschitz continuous if
$$\int_\Omega \|Tx(\omega) - Ty(\omega)\| dP(\omega) \leq \int_\Omega \kappa \|x(\omega) - y(\omega)\| dP(\omega), \quad \forall x(\omega), y(\omega) \in \mathfrak{H}$$
ii. Nonexpansive if (i) holds with $\kappa = 1$,
iii. η-strongly monotone if

$$\int_{\Omega} \langle Tx(\omega) - Ty(\omega), x(\omega) - y(\omega) \rangle dP(\omega) \geq \int_{\Omega} \eta \|x(\omega) - y(\omega)\| dP(\omega)$$

iv. Firlmy nonexpansive if

$$\int_{\Omega} \|Tx(\omega) - Ty(\omega)\|^2 dP(\omega) \leq \int_{\Omega} \langle Tx(\omega) - Ty(\omega), x(\omega) - y(\omega)\rangle dP(\omega)$$

v. α-averaged if $T = (1-\alpha)I + \alpha N$ for some fixed $\alpha \in (0,1)$ and nonexpansive mapping N. This, particularly, helps in expressing the operator, T as an averaged mapping, which is applied in the second part of the proof, to prove the existence of a fixed point solution

The conditions in this definition are applied in the approximation procedures, in the main results, especially, when proving boundednessof the iterative scheme.

Definition 2.4 (Udom, 2019)
Let $\{x(\omega)\}_{n=0}^{\infty}$ be a random sequence, and let $x(\omega)$ be a random point. The sequence $\{x(\omega)\}_{n=0}^{\infty}$ is said to converge in quadratic mean to $x(\omega)$ if $\lim_{n\to\infty} E\|x_n(\omega) - x(\omega)\|^2 = 0$. This is used in the last part of the proof, to show the convergence of the sequence generated by the algorithm, to a random fixed point, in expectation.

Definition 2.5 (Xu, 2010)
Let (Ω, Φ, P) be a complete probability measure space, C be a nonempty subset of a separable Hilbert space. The metric projection from $\mathfrak{H}$ onto C, denoted by Π_C is defined in such a way that, for each $x(\omega) \in \mathfrak{H}$, $\Pi_C\big(x(\omega)\big)$ is the unique point in C with the properties;

i. $\|x(\omega) - \Pi_C\big(x(\omega)\big)\| = \min\{\|x(\omega) - y(\omega)\| : y(\omega) \in C\}$
ii. Given $z(\omega) \in C$, then $z(\omega) = \Pi_C\big(x(\omega)\big)$ iff
$\langle x(\omega) - z(\omega), y(\omega) - z(\omega)\rangle \leq 0; \forall y(\omega) \in C$.

Lemma 2.1 (Buong, 2017)
Let $\mathfrak{H}$ be a real Hilbert space and let $G: \mathfrak{H} \to \mathfrak{H}$ be η-strongly monotone and κ-Lipschitz continuous. The mapping $I - t\mu G$, for each $\mu = \left(0, \frac{2\eta}{\kappa^2}\right)$, is contractive, with constant $1 - t\tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. This lemma is conspicuous in the first part of the proof; where it is shown that the algorithm used, is bounded.

Lemma 2.2 (Buong & Duong, 2011)
Let $\{x(\omega)\}_{n=0}^{\infty}$ and $\{z(\omega)\}_{n=0}^{\infty}$ be bounded sequences in a Banach space E, such that for $\beta_n \in [0,1]\ \forall n \geq 0$, $x_{n+1}(\omega) = (1-\beta_n)x_n(\omega) + \beta_n z_n(\omega)$ and $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.
Assume that $\limsup_{n\to\infty}(E\|z_{n+1}(\omega) - z_n(\omega)\| - E\|x_{n+1}(\omega) - x_n(\omega)\|) \leq 0$, then
$\lim_{n\to\infty} E\|x_n(\omega) - z_n(\omega)\| = 0$. This lemma is used in the second part of the proof, to show the existence of a fixed point solution.

Lemma 2.3 (Zhou & Wang, 2013)
Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of nonegative real numbers, satisfying the following conditions;
$a_{n+1} \leq (1-b_n)a_n + b_n c_n$, where $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ are sequences of real numbers such that
$b_n \in [0,1]$, $\sum_{n=0}^{\infty} b_n = \infty$ and $\limsup_{n\to\infty} c_n \leq 0$. Then $\lim_{n\to\infty} a_n = 0$. This lemma is used in the last part of the proof, to show convergence in quadratic mean.

Lemma 2.4 (Cegielski, 2012)
Let $T: C \times \Omega \to \mathfrak{H}$ be nonexpansive and $y(\omega) \in C$ be a weak convergence point of a sequence $\{x(\omega)\}_{n=0}^{\infty}$. If $\lim_{n\to\infty} E\|Tx_n(\omega) - x_n(\omega)\| = 0$, then $y(\omega) \in Fix(T)$. This is reffered to as the Demi-closedness. This is the lemma that is applied to show that the fixed point solution is actually in the feasible set of the problem

Theorem 2.1 (Cegielski, 2012)
Let X be a complete metric space, and a random operator $T: X \to X$ be a contraction. Then, T has exactly one fixed point $x^*(\omega) \in X$. The orbit $\{T^n x_n(\omega)\}_{n=0}^{\infty}$ converges to $x^*(\omega)$ with a rate of geometric progression. This theorem is known as the Banach's Fixed-Point Theorem. This theorem is fundamental in showing that the fixed point solution obtained is unique.

Assume that $x^*(\omega) \in \Gamma : Ax^*(\omega) \in Q$, where C and Q are closed convex subsets of real Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively, and Π_C and Π_Q are the metric projections on C and Q, respectively. It implies that $(I - \Pi_Q)Ax^*(\omega) = 0$, which then implies that $\gamma A^*(I - \Pi_Q)Ax^*(\omega) = 0, \gamma \in (0,1)$, hence the fixed point equation

$$(I - \gamma A^*(I - \Pi_Q)A)x^*(\omega) = x^*(\omega) \quad (2.1)$$

Requiring that $x^*(\omega) \in C$ leads to the consideration of the fixed point equation

$$\Pi_C(I - \gamma A^*(I - \Pi_Q)A)x^*(\omega) = x^*(\omega) \quad (2.2)$$

Proposition 2.1

Given $x^*(\omega) \in C$. Then, $x^*(\omega)$ solves the random form of (2), developed as

$$\text{Find } x^*(\omega) \in C : Ax^*(\omega) \in Q, \quad (2.3)$$

If and only if $x^*(\omega)$ solves the fixed point equation

$$x^*(\omega) = Tx^*(\omega); \quad (2.4)$$

where $T = \Pi_C(I - \gamma A^*(I - \Pi_Q)A)$

Proof:

By definition 2.6

$\langle (I - \gamma A^*(I - \Pi_Q)A)x^*(\omega) - x^*(\omega), z(\omega) - x^*(\omega)\rangle \le 0, z(\omega) \in C$

$\langle x^*(\omega) - \gamma A^*(I - \Pi_Q)Ax^*(\omega) - x^*(\omega), z(\omega) - x^*(\omega)\rangle \le 0$

$\langle -\gamma A^*(I - \Pi_Q)Ax^*(\omega),\ z(\omega) - x^*(\omega)\rangle \le 0$

$\langle A^*(I - \Pi_Q)Ax^*(\omega), z(\omega) - x^*(\omega)\rangle \ge 0$

$\langle (I - \Pi_Q)Ax^*(\omega), A\, z(\omega) - Ax^*(\omega)\rangle \ge 0$

$$\langle Ax^*(\omega) - \Pi_Q Ax^*(\omega), Ax^*(\omega) - Az(\omega)\rangle \le 0,\ z(\omega) \in C \quad (2.5)$$

Similarly,

$$\langle Ax^*(\omega) - \Pi_Q Ax^*(\omega), v(\omega) - Az(\omega)\rangle \le 0,\ v(\omega) \in Q \quad (2.6)$$

Adding (2.5) and (2.6) yields

$$\langle Ax^*(\omega) - \Pi_Q Ax^*(\omega), v(\omega) - Az(\omega)\rangle \le 0;\ z(\omega) \in C,\ v(\omega) \in Q \quad (2.7)$$

Substituting $z(\omega) = x^*(\omega) \in C$ and $v(\omega) = \Pi_Q Ax^*(\omega) \in Q$ results in

$\langle Ax^*(\omega) - \Pi_Q Ax^*(\omega), \Pi_Q Ax^*(\omega) - Az(\omega)\rangle \le 0$; which implies that

$$Ax^*(\omega) = \Pi_Q Ax^*(\omega) \in Q \quad (2.8)$$

So, $\Gamma = Fix(T) = C \cap A^{-1}Q$

Armed with these definitions, lemmas and preposition, it is time to prove the main result.

3. Results

3.1 Theorem 3.1

Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be two real Hilbert spaces, and C and Q be two closed convex subsets in $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively. Let A be a self-adjoining linear bounded mapping from $\mathfrak{H}_1$ into $\mathfrak{H}_2$. Let G be a strongly monotone and Lipschitz continuous mapping on $\mathfrak{H}_1$ with parameters η and κ, respectively. Assume that $\mu \in \left(0, \frac{2\eta}{\kappa^2}\right)$ is a fixed number. Let $\{x_n(\omega)\}_{n=0}^{\infty}$ be a random sequence of σ-algebra, adapted to the filtration Σ defined by

$$\begin{cases} y_n(\omega) = (1 - \rho_n)x_n(\omega) + \rho_n(I - t_n G)T^n x_n(\omega) \\ x_{n+1}(\omega) \ = \ (1 - \alpha_n)x_n(\omega) + \alpha_n(I - t_n\mu G)T^n y_n(\omega),\ \forall n \ge 1 \end{cases} \quad (3.1)$$

where $\rho_n \in (0,1)$; $\alpha_n \in (0,1)$; $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$ and t_n satisfies the conditions; (i) $t_n \in (0,1)$ and (ii) $\lim_{n\to\infty} t_n = 0$. Then, the random sequence, generated by (3.1), converges in quadratic mean to the unique random fixed point of T, which is the solution of (1.6).

Proof;

The proof to this theorem will be carried out in four cardinal steps;

i. Show that the sequence generated by (3.1) is bounded
ii. Show that algorithm given by (3.1) has a fixed point solution
iii. Show that the fixed point solution in (ii) is in the feasible set of the problem
iv. Show the convergence of the sequence generated by (3.1) to the unique solution of (1.6) in expectation.

(3.1) can be rewritten as;

$$x_{n+1}(\omega) \ = \ (1 - \alpha_n)x_n(\omega) + \alpha_n(I - t_n\mu G)T^n \ddot{T}^n x_n(\omega), \quad (3.2)$$

where $\ddot{T} := (1 - \rho_n)I + \rho_n(I - t_n\mu G)T$ and $T := \Pi_C(I - \gamma A^*(I - \Pi_Q)A)$.

Obviously, T is averaged; and so is $\ddot{T}$. Let $T_i : i = 1,2$ be β^i-averaged; then, T_1T_2 is β-averaged, where $\beta = \beta^1 + \beta^2 - \beta^1\beta^2$. Consequently, by definition (2.3-v), there exists a nonexpansive mapping, J, such that;

$$T\ddot{T} = (1 - \beta)I + \beta J \quad (3.3)$$

Step One involves showing that $\{x_n(\omega)\}_{n=0}^{\infty}$ is bounded.

Let $\ddot{x}(\omega)$ be any point in the closed convex subset, C, so that

$$E\|x_{n+1}(\omega)-\tilde{x}(\omega)\| = E\left\|(1-\alpha_n)(x_n(\omega)-\tilde{x}(\omega))+\alpha_n(I-t_n\mu G)T^n\ddot{T}^n x_n(\omega)-\tilde{x}(\omega)\right\|$$
$$= E\left\|(1-\alpha_n)(x_n(\omega)-\tilde{x}(\omega))+\alpha_n\{(I-t_n\mu G)T^n\ddot{T}^n x_n(\omega)-(I-t_n\mu G)\tilde{x}(\omega)+ (I-t_n\mu G)\tilde{x}(\omega)-\tilde{x}(\omega)\}\right\|$$
$$= E\left\|(1-\alpha_n)(x_n(\omega)-\tilde{x}(\omega))+\alpha_n\{(I-t_n\mu G)T^n\ddot{T}^n x_n(\omega)-(I-t_n\mu G)\tilde{x}(\omega)-t_n\mu G\tilde{x}(\omega)\}\right\|$$
$$\le (1-\alpha_n)E\|x_n(\omega)-\tilde{x}(\omega)\|+\alpha_n\{(1-\tau t_n)E\left\|T^n\ddot{T}^n x_n(\omega)-\tilde{x}(\omega)\right\|+t_n\mu E\|G\tilde{x}(\omega)\|\}$$
$$\le (1-\alpha_n)E\|x_n(\omega)-\tilde{x}(\omega)\|+\alpha_n\{(1-\tau t_n)E\|x_n(\omega)-\tilde{x}(\omega)\|+t_n\mu E\|G\tilde{x}(\omega)\|\}$$
$$= (1-\alpha_n+\alpha_n-\tau\alpha_n t_n)E\|x_n(\omega)-\tilde{x}(\omega)\|+\alpha_n t_n\mu E\|G\tilde{x}(\omega)\|$$
$$= (1-\tau\alpha_n t_n)E\|x_n(\omega)-\tilde{x}(\omega)\|+\tau\alpha_n t_n\frac{\mu}{\tau}E\|G\tilde{x}(\omega)\|$$
$$\le (1-\tau\alpha_n t_n)S+\tau\alpha_n t_n S = S$$

where $S = max\{E\|x_0(\omega)-\tilde{x}(\omega)\|, \frac{\mu}{\tau}E\|G\tilde{x}(\omega)\|\}$ is a fixed constant.

Then $E\|x_0(\omega)-\tilde{x}(\omega)\| \le S$, which implies that $E\|x_n(\omega)-\tilde{x}(\omega)\| \le S;\ \forall n \ge 0$

Hence, $E\|x_{n+1}(\omega)-\tilde{x}(\omega)\| \le S$, which means that $\{x_n(\omega)\}_{n=0}^{\infty}$ is bounded. Also, $\{T^n\ddot{T}^n x_n(\omega)\}_{n=0}^{\infty}$ and $\{G[T^n\ddot{T}^n x_n(\omega)]\}_{n=0}^{\infty}$ are bounded. Having shown that the generated sequence is bounded, it is time to show that the algorithm has a fixed point solution.

Step Two is to show that $\lim_{n\to\infty} E\|x_{n+1}(\omega)-x_n(\omega)\| = 0$

Let $(I-t_n\mu G)T^n\ddot{T}^n x_n(\omega) = z_n(\omega)$.

Then,

$$E\|z_{n+1}(\omega)-z_n(\omega)\| = E\left\|(I-t_{n+1}\mu G)T^{n+1}\ddot{T}^{n+1}x_{n+1}(\omega)-(I-t_n\mu G)T^n\ddot{T}^n x_n(\omega)\right\|$$
$$= E\left\|T^{n+1}\ddot{T}^{n+1}x_{n+1}(\omega)-T^n\ddot{T}^n x_n(\omega)+t_n\mu G[T^n\ddot{T}^n x_n(\omega)]-t_{n+1}\mu G\left[T^{n+1}\ddot{T}^{n+1}x_{n+1}(\omega)\right]\right\|$$
$$= E\left\|T^{n+1}\ddot{T}^{n+1}x_{n+1}(\omega)-T^{n+1}\ddot{T}^{n+1}x_n(\omega)+T^{n+1}\ddot{T}^{n+1}x_n(\omega)-T^n\ddot{T}^n x_n(\omega)+t_n\mu G[T^n\ddot{T}^n x_n(\omega)] - t_{n+1}\mu G\left[T^{n+1}\ddot{T}^{n+1}x_{n+1}(\omega)\right]\right\|$$
$$\le E\left\|T^{n+1}\ddot{T}^{n+1}x_{n+1}(\omega)-T^{n+1}\ddot{T}^{n+1}x_n(\omega)\right\|+E\left\|T^{n+1}\ddot{T}^{n+1}x_n(\omega)-T^n\ddot{T}^n x_n(\omega)\right\|+t_n\mu E\left\|G[T^n\ddot{T}^n x_n(\omega)]\right\| + t_{n+1}\mu E\left\|G\left[T^{n+1}\ddot{T}^{n+1}x_{n+1}(\omega)\right]\right\|$$

$\le E\|x_{n+1}(\omega)-x_n(\omega)\|+E\left\|T^{n+1}\ddot{T}^{n+1}x_n(\omega)-T^n\ddot{T}^n x_n(\omega)\right\|+t_n\mu E\left\|G[T^n\ddot{T}^n x_n(\omega)]\right\|+ t_{n+1}\mu E\left\|G\left[T^{n+1}\ddot{T}^{n+1}x_{n+1}(\omega)\right]\right\|$ (the first term is because $T\ddot{T}$ is contractive).

So that, moving the first term on the right, to the left side of the inequality results in;

$$E\|z_{n+1}(\omega)-z_n(\omega)\|-E\|x_{n+1}(\omega)-x_n(\omega)\| \le E\left\|T^{n+1}\ddot{T}^{n+1}x_n(\omega)-T^n\ddot{T}^n x_n(\omega)\right\|+t_n\mu E\left\|G[T^n\ddot{T}^n x_n(\omega)]\right\|+ t_{n+1}\mu E\left\|G\left[T^{n+1}\ddot{T}^{n+1}x_{n+1}(\omega)\right]\right\| \quad (3.4)$$

By definition (2.3-v), there exists $\nu \in (0,1)$ such that $T^n\ddot{T}^n = (1-\nu_n)I+\nu_n N$, so that the second term of (3.4) is expressed and simplified as

$$E\left\|T^{n+1}\ddot{T}^{n+1}x_n(\omega)-T^n\ddot{T}^n x_n(\omega)\right\| = E\|\{(1-\nu_{n+1})I+\nu_{n+1}N\}x_n(\omega)-\{(1-\nu_n)I+\nu_n N\}x_n(\omega)\|$$
$$= E\|x_n(\omega)-\nu_{n+1}x_n(\omega)+\nu_{n+1}Nx_n(\omega)-x_n(\omega)+\nu_n x_n(\omega)-\nu_n Nx_n(\omega)\|$$
$$= E\|\nu_{n+1}[Nx_n(\omega)-x_n(\omega)]-\nu_n[Nx_n(\omega)-x_n(\omega)]\|$$
$$= E\|(\nu_{n+1}-\nu_n)[Nx_n(\omega)-x_n(\omega)]\|$$
$$= E\|(\nu_{n+1}-\nu_n)Nx_n(\omega)-(\nu_{n+1}-\nu_n)x_n(\omega)\|$$
$$\le (\nu_{n+1}-\nu_n)E\|Nx_n(\omega)\|+(\nu_{n+1}-\nu_n)E\|x_n(\omega)\|$$
$$= (\nu_{n+1}-\nu_n)\{E\|Nx_n(\omega)\|+E\|x_n(\omega)\|\}$$
$$\le (\nu_{n+1}-\nu_n)K$$

where $K = \sup_{n\ge 0}\{E\|Nx_n(\omega)\|+E\|x_n(\omega)\|\}$; and (3.4) becomes;

$$E\|z_{n+1}(\omega)-z_n(\omega)\|-E\|x_{n+1}(\omega)-x_n(\omega)\| \le (\nu_{n+1}-\nu_n)K+t_n\mu E\left\|G[T^n\ddot{T}^n x_n(\omega)]\right\|+ t_{n+1}\mu E\left\|R\left[T^{n+1}\ddot{T}^{n+1}x_{n+1}(\omega)\right]\right\| \quad (3.5)$$

Taking the limits of both sides of (3.5) yields;

$$\lim_{n\to\infty}\sup(E\|z_{n+1}(\omega)-z_n(\omega)\|-E\|x_{n+1}(\omega)-x_n(\omega)\|) \le 0\,; \quad (3.6)$$

because, $\lim_{n\to\infty}(\nu_{n+1}-\nu_n)K = 0$ and $\lim_{n\to\infty} t_n = \lim_{n\to\infty} t_{n+1} = 0$ (by the conditions in theorem 3.1)

By lemma (2.2) and (3.6), $\lim_{n\to\infty} E\big(z_n(\omega)-x_n(\omega)\big) = 0$; and by (3.1)

$\lim_{n\to\infty} E\big(x_{n+1}(\omega)-x_n(\omega)\big) = 0$, which also implies

$$\lim_{n\to\infty} E\left(x_n(\omega)-T^n\ddot{T}^n x_n(\omega)\right) = 0 \quad (3.7)$$

By (3.7) it is seen that the algorithm has a fixed point solution. That is;

$$\lim_{n\to\infty} E\big(x_n(\omega)\big) = \lim_{n\to\infty} E\left(T^n\ddot{T}^n x_n(\omega)\right)$$

Step Three shows that $\limsup_{n\to\infty}\langle E(R[x^*(\omega)]), x^*(\omega) - x_n(\omega)\rangle \le 0$ by the demi-closedness principle.
Let $x^*(\omega) \in H$ be the unique solution of (1.6); and let $\{x_{nr}(\omega)\}_{n=1}^{\infty}$ be a subsequence of $\{x_n(\omega)\}_{n=0}^{\infty}$, that converges to a random fixed point $\acute{x}(\omega)$ as $n \to \infty$, such that;

$$\limsup_{n\to\infty}\langle E(G[x^*(\omega)]), x^*(\omega) - T^n\ddot{T}^n x_n(\omega)\rangle = \limsup_{n\to\infty}\langle E(G[x^*(\omega)]), x^*(\omega) - x_n(\omega)\rangle$$
$$\limsup_{n\to\infty}\langle E(G[x^*(\omega)]), x^*(\omega) - \acute{x}(\omega)\rangle$$

Since it is expected that $\acute{x}(\omega) \in C$,

$$\limsup_{r\to\infty}\langle E(G[x^*(\omega)]), x^*(\omega) - \acute{x}(\omega)\rangle \le 0$$

It is left to show that $\acute{x}(\omega) \in C$
Assume that $\psi_{nr} \to \psi_\infty$, as $r \to \infty$; $0 < \psi' \le \psi_\infty \le \psi'' < 1$
By definition (2.3-v), $T^\infty = (1-\psi_\infty)I + \psi_\infty$, so that $Fix(T^\infty) = Fix(T) \neq \emptyset$ and

$$\limsup_{r\to\infty,\ x\in D} E\|T^{nk}x(\omega) - T^\infty x(\omega)\| = 0 \qquad (3.8)$$

where D is an arbitrary bounded subset containing $\{x_n(\omega)\}_{n=0}^{\infty}$.
By (3.7) and (3.8), it is obtained that $\lim_{r\to\infty} E\left(x_{nr}(\omega) - T^\infty x_{nr}(\omega)\right) = 0$
From the demi-closedness principle in lemma 2.4, $\acute{x}(\omega) \in C$, as required.

Finally, it is left to show that $\lim_{n\to\infty} E\|x_n(\omega) - x^*(\omega)\|^2 = 0$

$$E\|x_{n+1}(\omega) - x^*(\omega)\|^2 = E\left\|(1-\alpha_n)(x_n(\omega) - x^*(\omega)) + \alpha_n(I - t_n\mu G)T^n\ddot{T}^n x_n(\omega) - x^*(\omega)\right\|^2$$
$$= E\left\|(1-\alpha_n)(x_n(\omega) - x^*(\omega)) + \alpha_n\left[(I - t_n\mu G)T^n\ddot{T}^n x_n(\omega) - (I - t_n\mu G)x^*(\omega) + (I - t_n\mu G)x^*(\omega) - x^*(\omega)\right]\right\|^2$$
$$= E\left\|(1-\alpha_n)(x_n(\omega) - x^*(\omega)) + \alpha_n\left[(I - t_n\mu G)T^n\ddot{T}^n x_n(\omega) - (I - t_n\mu G)x^*(\omega) - t_n\mu G x^*(\omega)\right]\right\|^2$$
$$\le (1-\alpha_n)E\|x_n(\omega) - x^*(\omega)\|^2 + \alpha_n E\left\|(1-\tau t_n)\left(T^n\ddot{T}^n x_n(\omega) - x^*(\omega)\right) - t_n\mu G x^*(\omega)\right\|^2$$
$$= (1-\alpha_n)E\|x_n(\omega) - x^*(\omega)\|^2 + \alpha_n E\left\langle \begin{matrix}(1-\tau t_n)\left(T^n\ddot{T}^n x_n(\omega) - x^*(\omega)\right) - t_n\mu G x^*(\omega), \\ (1-\tau t_n)\left(T^n\ddot{T}^n x_n(\omega) - x^*(\omega)\right) - t_n\mu G x^*(\omega)\end{matrix}\right\rangle$$
$$\le (1-\alpha_n)E\|x_n(\omega) - x^*(\omega)\|^2 + \alpha_n E\{(1-\tau t_n)\left\|T^n\ddot{T}^n x_n(\omega) - x^*(\omega)\right\|^2$$
$$+2t_n\mu\langle Gx^*(\omega), T^n\ddot{T}^n x_n(\omega) - x^*(\omega)\rangle + t_n^2\mu^2\langle Gx^*(\omega), G\left[T^n\ddot{T}^n x_n(\omega)\right]\rangle\}$$
$$\le (1-\alpha_n)E\|x_n(\omega) - x^*(\omega)\|^2 + \alpha_n(1-\tau t_n)E\|x_n(\omega) - x^*(\omega)\|^2$$
$$+2\alpha_n t_n\mu E\langle Gx^*(\omega), T^n\ddot{T}^n x_n(\omega) - x^*(\omega)\rangle + \alpha_n t_n^2\mu^2 E\langle Gx^*(\omega), G\left[T^n\ddot{T}^n x_n(\omega)\right]\rangle$$
$$= (1-\alpha_n\tau t_n)E\|x_n(\omega) - x^*(\omega)\|^2 + 2\alpha_n t_n\mu E\langle Gx^*(\omega), T^n\ddot{T}^n x_n(\omega) - x^*(\omega)\rangle$$
$$+\alpha_n t_n^2\mu^2 E\langle Gx^*(\omega), G\left[T^n\ddot{T}^n x_n(\omega)\right]\rangle$$
$$= (1-\tau\alpha_n t_n)E\|x_n(\omega) - x^*(\omega)\|^2 + \tau\alpha_n t_n\{\frac{2\mu}{\tau}E\langle Gx^*(\omega), T^n\ddot{T}^n x_n(\omega) - x^*(\omega)\rangle$$
$$+\frac{t_n\mu^2}{\tau}E\langle Gx^*(\omega), G\left[T^n\ddot{T}^n x_n(\omega)\right]\rangle\}$$
$$= (1-\tau\alpha_n t_n)E\|x_n(\omega) - x^*(\omega)\|^2 + \tau\alpha_n t_n\{\frac{2\mu}{\tau}E\langle Gx^*(\omega), T^n\ddot{T}^n x_n(\omega) - x^*(\omega)\rangle$$
$$+\frac{t_n\mu^2}{\tau}E\|Gx^*(\omega)\|\left\|G\left[T^n\ddot{T}^n x_n(\omega)\right]\right\|$$
$$\le (1-\tau\alpha_n t_n)E\|x_n(\omega) - x^*(\omega)\|^2 + \tau\alpha_n t_n\{\frac{2\mu}{\tau}E\langle Gx^*(\omega), T^n\ddot{T}^n x_n(\omega) - x^*(\omega)\rangle$$
$$+\frac{t_n\mu^2}{\tau}E\|Gx^*(\omega)\|M\}$$

where $M \ge \sup_{n\ge 0} E\left\|G\left[T^n\ddot{T}^n x_n(\omega)\right]\right\|$ is a fixed constant.

$$E\|x_{n+1}(\omega) - x^*(\omega)\|^2 \le (1-\tau\alpha_n t_n)E\|x_n(\omega) - x^*(\omega)\|^2 + \tau\alpha_n t_n c_n \qquad (3.9)$$

where $c_n = \frac{2\mu}{\tau}E\langle Gx^*(\omega), T^n\ddot{T}^n x_n(\omega) - x^*(\omega)\rangle + \frac{t_n\mu^2}{\tau}E\|Gx^*(\omega)\|M$.
Clearly, $\limsup_{n\to\infty} c_n \le 0$, because $\limsup_{n\to\infty} E\langle Gx^*(\omega), x^*(\omega) - x_n(\omega)\rangle \le 0$, $\lim_{n\to\infty} t_n = 0$ and

$$\lim_{n\to\infty} E\left\|x_n(\omega) - T^n\ddot{T}^n x_n(\omega)\right\| = 0$$

Comparing (3.9) with lemma 2.3, it is concluded that;
$\lim_{n\to\infty} E\|x_n(\omega) - x^*(\omega)\|^2 = 0$, which implies that $\{x_n(\omega)\}_{n=0}^{\infty}$, converges in quadratic mean to $x^*(\omega)$.
It is observed that $x^*(\omega) \in Fix(T)$, and by proposition 1, $Fix(T)$ is the same as Γ, which is the feasible set of the problem. Hence, $x^*(\omega)$ is the unique solution to (1.6), and the proof is completed.

Remark

The solution to the stochastic variational inequality problem, using the iteration belong to the fixed point set of the linear operator, T denoted by $Fix\ (T)$; and from proposition 1, the solution set of the stochastic split feasibility problem, Γ coincides with $Fix(T)$. That is $\Gamma = C \cap A^{-1}Q = Fix(T)$. By Theorem 2.1, a unique solution $x^*(\omega) \in Fix(T)$ is guaranteed. This solution satisfies both the stochastic variational and the stochastic split feasibility problems; thereby solving the random split variational inequality problem in (1.6).

3.2 Numerical Illustration of Result

Suppose that a government classifies businesses in the state into two categories – low earning businesses, and high earning businesses; and wishes to find an optimal tax policy that would:

i. Meet revenue target,
ii. Minimize tax burden on low-income businesses,
iii. Encourage economic growth

The problem is defined and formulated as;

a) The two groups of businesses, low (1) and high (2), have the tax rates, respectively, denoted by $x_l(\omega)$ and $x_h(\omega)$.
b) Let the targeted revenue be denoted by L.
c) Suppose there are n_l low earning businesses, with average income, I_l; and n_h high earning businesses, with average income, I_h.
d) Let ϕ_l and ϕ_h respectively denote the maximum feasible tax rates for the low and high earning businesses.

The following Objective function, which is the tax component of Gross Domestic Production (GDP), is:
$Max\, W = \gamma_0 + \gamma_l(1 - x_l(\omega)) + \gamma_h\big(1 - x_h(\omega)\big) + \varepsilon,$
subject to the following constraints:
$\sum_{i=1}^{2} \psi_i x_i(\omega) = L$ (Revenue constraint)
$0 \le x_l(\omega) \le \phi_l, 0 \le x_h(\omega) \le \phi_h$ (Tax rate constraint);
where;

- γ_0 is the baseline growth rate
- γ_l is the impact of low earning tax rate on GDP growth
- γ_h is the impact of high earning tax rate on GDP growth
- ε is the random error associated with W

The random SFP is expected to find
$\boldsymbol{x}^*(\omega) \in C: A\boldsymbol{x}(\boldsymbol{\omega}) \in Q$
where $C = \{(x_l^*(\omega), x_h^*(\omega)): 0 \le x_l \le \phi_1, 0 \le x_h \le \phi_2\}$
$A = (\psi_1\ \psi_2)$
$Q = \{(x_l^*(\omega), x_h^*(\omega)): A(x_l^*(\omega), x_h^*(\omega)) \ge L$

The mapping A takes tax rates from C, the feasible set, and produces revenue with the tax rates that meet the revenue target.
The random VIP, as formulated in (3.1), has the monotone, Lipschitz function, G, as the negative gradient of W.
The random VIP is to provide equilibrium for tax payers. It ensures that the tax rates are optimal. It also ensures that tax policy is fair and acceptable to both groups.

Suppose there are 2,500 low earning businesses, with average monthly income of ₦200,000, and 800 high earning businesses earners, with average income of ₦1,000,000; and that the government is interested in raking in ₦ 150,000,000, from these businesses. Let the maximum tax rates for these groups of earnings be, respectively, 15% and 27%.
Giving real values to the parameters of the function, W, assume;
$Max\, W = 0.5 + 0.065(1 - x_l(\omega)) + 0.03\big(1 - x_h(\omega)\big) + \varepsilon$
Subject to $0 \le x_l(\omega) \le 0.15$
$0 \le x_h(\omega) \le 0.27$
$500{,}000{,}000 x_l(\omega) + 800{,}000{,}000 x_l(\omega) \ge 150{,}000{,}000,$
where: $500{,}000{,}000 = 2{,}500{,}000 \times 200{,}000$ and $800{,}000{,}000 = 800 \times 1{,}000{,}000$.
The function, W, for the random VIP, is the negative gradients of the GDP growth function.
That is;
$G(x_l(\omega), x_h(\omega)) = \left(-\frac{\partial W(x_l(\omega), x_h(\omega))}{\partial x_l}, -\frac{\partial W(x_l(\omega), x_h(\omega))}{\partial x_h}\right) = (0.065, 0.03).$
Let $A = \begin{pmatrix} 0.84 & 0 \\ 0 & 0.71 \end{pmatrix}$ and its adjoint $A^* = \begin{pmatrix} 0.84 & 0 \\ 0 & 0.71 \end{pmatrix}$. Also, let $B = A^* \times A$
Then, the linear operator for the random SFP, is given as;
$B\big(x(\omega)\big) = \begin{pmatrix} 0.7 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} x_l(\omega) \\ x_h(\omega) \end{pmatrix}.$
Setting both the Lipschitzian and monotonicity coefficients to 1, then;
$\mu \in \left(0, \frac{2\eta}{\kappa^2}\right)$ becomes $\mu \in (0, 2)$.
The iterative solution which is summarized in the Table 1, is carried out using R software (version 4.5.0).

Table 1: Summary of iterations

Iterations	x_l	x_h
1	0.102058	0.101516
2	0.104083	0.103000
3	0.106074	0.104452
4	0.108033	0.105874
5	0.109959	0.107266
6	0.111853	0.108628
7	0.113716	0.109963
8	0.115547	0.111269
9	0.117347	0.112548
10	0.119116	0.113801
The last ten iterations	0.144132	0.131586
	0.144889	0.132461
	0.145524	0.133323
	0.146059	0.134175
	0.146507	0.135017
	0.146884	0.135851
	0.147201	0.136676
	0.147467	0.137494
	0.147690	0.138305
	0.147878	0.139110
Number of iterations	36	36

$\varepsilon \sim N(0,1)$

4. Discussion

It is proven that the iterative algorithm in 3.1 generates a sequence that is bounded; and the algorithm has a random fixed point solution. This is a point where the input of the algorithm is the same as the output. Furthermore, it is shown that this fixed point solution is in the feasible set of the problem. Then, a strong convergence result is obtained for the solution, since convergence in quadratic mean is a stronger mode of convergence, when compared to convergence in probability and convergence in distribution. This result extends and generalizes that of Miao & Li (2008), which was, not only carried out in the deterministic case, but also only proved a weak convergence result.

In illustrating the result numerically, a hypothetical government tax policy optimization problem is formulated and solved, using the result. The concept of fixed point solution is further exemplified here. Since the value (output) of a preceding iteration step serves as input to the succeeding iteration step, it is observed, from Table 1, that the difference between the last two iteration values is quite negligible; which means that the input is the same as the output. The result suggests the optimum tax rates for the low and high earning businesses to be 14.79% and 13.91%, respectively. Ordinarily, this result looks odd – that the low earning businesses is expected to pay a higher tax rate than the high earning businesses. However, the result would not be so out of place, considering the fact that the tax values are meant to satisfy two conditions – meeting the revenue target, which is the random SFP component of the problem; and contributing, maximally, to the GDP growth, which is the random VIP part of the problem. For instance, substituting these tax rates into the revenue constraint, results in ₦ 185,227,000 which satisfies the constraint; at the same time, since it is the non-taxable income that goes back to circulation, thereby boosting the GDP, the expected amounts that go back to circulation are 85.21% of 500,000,000 and 86.09% of 800,000,000, for the low and high earning businesses, respectively.

It is also worthy of mention that the Lipschitzian condition is a rare property, as not so many functions possess it. When this condition is removed, it might be difficult to obtain convergence, if not outrightly impossible – here lies its major limitation.

5. Conclusion

In this paper, convergence result in classical functional analysis is extended, to develop convergence result of the solution of random split variational inequality problem, based on the random two-step Wang's algorithm. It is shown that the solution converges in quadratic mean to a unique random fix point in a Hilbert space. Furthermore, an application of the result to optimal tax policy problem is presented. In the application, optimum tax rates have been obtained: 14.79% and 13.91%, respectively for the low and high earning businesses.

The work done here can be further extended by carrying out the same study in a more general Banach space, where the property of inner product is relaxed; using other types of operators, like the psuedocontractive operators; and trying out other functions which are not monotonic.

Declaration of Interest

The authors report there are no competing interest to disclose.

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