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Structural and Variance Properties of Multivariate Autoregressive Models: Empirical Applications to Nigerian Economic Variables

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Abstract

Multivariate Autoregressive models are widely used models in the study of economic variables. However, the understanding of the mean and variance parameters of a Multivariate Autoregressive Model enhances the accuracy and reliability of statistical inferences, forecasting precision, and model stability. It provides deeper insights into the model's dynamic behavior, allowing for better decision-making in economic, financial, and epidemiological applications. Additionally, it aids in selecting the most appropriate model for specific data structures, ensuring robust and consistent predictions. Using the Nigerian macro-economic variables, the paper advances the theoretical and practical understanding of Multivariate Autoregressive Distributed Lag (MARDL) models by systematically analyzing their properties in comparison to Vector Autoregressive (VAR) models. This paper evaluates the mean and variance properties of both models to assess their relative performance. Findings indicate no significant difference in their mean properties; however, a variance analysis reveals that the MARDL model exhibits smaller variances compared to the VAR model. This suggests that the MARDL model offers greater stability, control, and consistency, making it a more reliable tool for prediction and forecasting. The results contribute to the ongoing refinement of multivariate time series modeling and its application in empirical research.

Keywords: VAR, MARDL, Mean, and Variance.

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Introduction

Vector Autoregressive (VAR) models are an extension of univariate time series models, incorporating multiple response variables as functions of their own lagged terms and the lagged terms of the other variables. These models operate on both a feed-forward and feedback mechanism, where each response variable in a VAR models is a linear combination of its own lags, the lags of other predictors and an error term, resembling a multivariate form of linear regression. This structure allows VAR models to capture the interdependence and dynamic interactions among variables. This makes them particularly suitable for forecasting economic and financial time series data, gaining significant prominence following the seminal work of Sims (1980)

Research on the properties of Vector Autoregressive (VAR) models has been extensive, particularly in improving mean and variance estimation for the analysis of economic and financial data. A central objective across studies has been to enhance the accuracy of variance estimations, aiming to better capture the dynamics of high-dimensional and structurally complex multivariate systems.

Sims (1982) critically examined the limitations of variance decomposition in policy-oriented VAR models, emphasizing challenges in interpreting results, especially in the presence of unit roots and unstable variances. Similarly, Bollerslev, Engle, and Wooldridge (1988) explored time-varying variances in larger systems but highlighted issues of over-parameterization in high-dimensional applications. These authors further stressed the implications of time-varying variance for financial models, advocating for parsimonious approaches in multi-volatility regime systems. Sims and Uhlig (1991) extended this discourse by emphasizing the difficulties posed by nonstationary boundaries, which complicate accurate variance estimation. Building on these findings, Balke and Fomby (1997) introduced threshold VAR models to account for nonlinear regime changes. However, they noted persistent challenges in selecting appropriate thresholds.

In a related critique, Canova and De Nicolo (2002) highlighted the limitations of traditional VAR models in addressing variance spillovers during international business cycles, thereby exposing a gap in accounting for global economic interdependencies. Similarly, Blanchard and Perotti (2002) focused on the impact of fiscal policy, noting asymmetries in policy-induced variance. They observed significant challenges in representing lagged variance effects, underscoring the limitations of conventional models in capturing asymmetrical variance patterns for fiscal policy analysis.

Expanding on this discourse, Hamilton and Herrera (2004) examined oil price shocks within VAR frameworks. They emphasized the asymmetric responses of variance to positive and negative shocks, advocating for nonlinear specifications to better capture variance dynamics in volatile markets. Enders (2004) further addressed the issue of nonstationary variances, highlighting the forecasting challenges posed by structural breaks in both mean and variance. Hamilton and Herrera (2004) reiterated the gaps in conventional VAR models, particularly in handling nonlinear responses to variance dynamics caused by oil price shocks.

Uhlig (2005) applied structural vector autoregressions (SVARs) to monetary policy analysis, highlighting that variance decompositions can become unstable under certain identification constraints. Building on this, Primiceri (2005) extended vector autoregressive (VAR) models by incorporating time-varying variances and covariances, which enhanced their applicability in policy analysis. However, this advancement introduced interpretive challenges due to the added complexity.

Similarly, Tsay (2005) examined high-frequency time series with volatility shifts, underscoring the need for adaptive variance methods in VAR models to accurately represent dynamic variance patterns. In the same vein, Primiceri (2005) emphasized the empirical limitations of VAR models when analyzing high-dimensional datasets, despite their advancements in addressing time-varying structures.

Further extending the field, Lanne and Saikkonen (2007) proposed methods for handling conditional heteroskedasticity, identifying significant limitations of VAR models in managing abrupt structural changes. Meanwhile, Francis and Ramey (2009) tackled the challenges posed by low-frequency data, demonstrating how it could distort variance implications in fast-moving market contexts.

Expanding the scope, Forni, Giannone, Lippi, and Reichlin (2009) applied structural factor models to large cross-sectional datasets. However, they found that variance decomposition became increasingly difficult when latent factors remained unobservable. Similarly, Canova and Ciccarelli (2009) extended VARs to multi-country applications, revealing cross-sectional variance dependencies while grappling with country-specific heterogeneity.

Carriero, Clark, and Marcellino (2015) conducted a detailed investigation into Bayesian VAR specification choices, emphasizing that improper priors can significantly distort mean and variance outcomes, particularly in smaller samples. Building on this, Giannone, Lenza, and Primiceri (2015) introduced the integration of DSGE priors within Bayesian VARs to enhance variance estimation for macroeconomic forecasting. Similarly, Barigozzi, Lippi, and Luciani (2016) delved into network-based mean and variance structures in high-dimensional VAR models; however, their approach requires further refinement to ensure greater accuracy in variance decomposition for interconnected systems.

In the context of structural VARs (SVARs), Kilian and Lütkepohl (2017) highlighted persistent identification challenges, especially in cases of conditional heteroskedasticity. Their findings underscored the importance of developing adaptive identification techniques to improve the robustness of variance analysis. Along these lines, Arias, Rubio-Ramirez, and Waggoner (2018) explored the use of sign and zero restrictions to enhance the robustness of SVAR variance decomposition. Nonetheless, they noted that restrictive assumptions often limit the reliability of inference.

Stock and Watson (2018) further contributed to the discussion by examining the use of external instruments within VARs to improve causal identification of variance effects. Despite their potential, these instruments face challenges related to reliability, particularly in macroeconomic contexts. Also, Nakamura and Steinsson (2018) proposed adjustments to conventional VAR assumptions to better account for high-frequency data, identifying notable gaps in capturing unique variance dynamics at shorter time scales. In all of these, the model estimations of the Vector Autoregressive Model were lacking in the parameter estimates which this research seeks to address.

The Multivariate Autoregressive Distributed Lag (MARDL) model is a powerful tool for analyzing dynamic interrelationships among time series variables, capturing both short-term dynamics and long-term equilibrium relationships. It has become a cornerstone in various fields, including economics, environmental science, and social studies. This literature review synthesizes key studies that have applied MARDL, focusing on its strengths and limitations in estimating mean and variance relationships.

Multivariate Autoregressive Distributed Lag (MARDL) models are designed for multiple response variables, incorporating both lagged and non-lagged terms of predictor variables, while relying solely on the lagged terms of the response variables. In multivariate time series analysis, each variable is expressed as a linear combination of its lagged terms and those of other variables. Usoro (2019) developed MARDL models as a synthesis of Multivariate Linear Regression (MLR) and Vector Autoregressive (VAR) models. Multivariate Linear Regression (MLR) models describe a linear relationship between the current values of the response and predictor variables, whereas VAR models are commonly used in multivariate time series analysis and are characterized by autoregressive processes, it represents an extension of univariate time series models, where the response variable is a function of its own lagged terms.

A key distinction between MARDL and VAR models is that MARDL incorporates the present values of the predictor variables, while VAR models limit the independent variables to lagged terms. This distinction highlights the inherent causal relationship between the current values of predictor variables and the response variables.

Usoro (2020) employed cross-autocorrelation and cross-autocorrelation matrices to investigate the stability of multivariate time series. The analysis demonstrated that, in instances of instability, the overall stability of the process is influenced by partial stationarity. Building on this, Usoro and Udoh (2021) utilized the Multivariate Autoregressive Distributed Lag (MARDL) model to examine Nigeria's Gross Domestic Product alongside other macroeconomic indicators. Their findings highlighted a complementary relationship between the MARDL and Vector Autoregressive (VAR) models, underscoring the relevance of both methodologies in capturing economic dynamics. Despite its potential, the MARDL model remains underutilized, necessitating further research to establish its theoretical properties and practical effectiveness, which forms the core objective of this study.

Vector Autoregressive (VAR) Model

Definition

Let $\underline{Z}_t = (Z_{1t}, Z_{2t}, \dots, Z_{mt})'$ be the vector of response time variables, $\underline{\emptyset} = (\emptyset_{k.ij})$ is the vector of coefficients, $\underline{Z}_{t-k} = (Z_{1t-k}, Z_{2t-k}, \dots, Z_{nt-k})'$ be defined as the vector of the predictive lag time variables, $\underline{\delta} = (\delta_1, \delta_2, \dots, \delta_m)^I$ is the vector of constants and $\underline{w}_t = (w_{t1}, w_{2t}, \dots, w_{mt})'$ is the vector of error terms associated with the response time variables. The Vector Autoregressive Model is presented in the form, $\underline{Z}_t = \underline{\delta} + \emptyset\underline{Z}_{t-k} + \underline{w}_t$, expressed in a matrix form as $(Z_{1t}, Z_{2t}, \dots, Z_{mt})' = (\delta_1, \delta_2, \dots, \delta_m)' + (\emptyset_{k.ij})(Z_{1t-k}, Z_{2t-k}, \dots, Z_{nt-k})' + (w_{t1}, w_{2t}, \dots, w_{mt})'$ (3.1)

Expansion of Equation (3.1) gives

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \\ \vdots \\ Z_{mt} \end{pmatrix} = \begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} + \begin{pmatrix} \emptyset_{1.11} & \emptyset_{1.12} & \cdots & \emptyset_{1.1n} \\ \emptyset_{1.21} & \emptyset_{1.22} & \cdots & \emptyset_{1.2n} \\ \vdots & \vdots & \vdots & \vdots \\ \emptyset_{1.m1} & \emptyset_{1.m2} & \cdots & \emptyset_{1.mn} \end{pmatrix} \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \\ \vdots \\ Z_{mt-1} \end{pmatrix} + \begin{pmatrix} \emptyset_{2.11} & \emptyset_{2.12} & \cdots & \emptyset_{2.1n} \\ \emptyset_{2.21} & \emptyset_{2.22} & \cdots & \emptyset_{2.2n} \\ \vdots & \vdots & \vdots & \vdots \\ \emptyset_{2.m1} & \emptyset_{2.m2} & \cdots & \emptyset_{2.mn} \end{pmatrix} \begin{pmatrix} Z_{1t-2} \\ Z_{2t-2} \\ \vdots \\ Z_{mt-2} \end{pmatrix} + \dots$$

$$+ \begin{pmatrix} \emptyset_{p.11} & \emptyset_{p.12} & \cdots & \emptyset_{p.1n} \\ \emptyset_{p.21} & \emptyset_{p.22} & \cdots & \emptyset_{p.2n} \\ \vdots & \vdots & \vdots & \vdots \\ \emptyset_{p.m1} & \emptyset_{p.m2} & \cdots & \emptyset_{p.mn} \end{pmatrix} \begin{pmatrix} Z_{1t-p} \\ Z_{2t-p} \\ \vdots \\ Z_{mt-p} \end{pmatrix} + \begin{pmatrix} w_{1t} \\ w_{2t} \\ \vdots \\ w_{mt} \end{pmatrix} \quad (3.2)$$

More simplified form of Equation (3.2) by Gujarati and Porter (2009) is

$$Z_{it} = \delta_i + \sum_{k=1}^{p}\sum_{j=1}^{n} \emptyset_{k.ij} Z_{jt-k} + w_{it},\ i = 1, \dots, m, Where\ i = 1, 2, \dots, m; j = 1, 2, \dots, n; k = 1, 2, \dots, p \quad (3.3)$$

Model Expansion: Given Equation (3.2)

Case 1:

$if\ i = 1; j = 1, \dots, n; k = 1, \dots, p$

$$Z_{1t} = \delta_1 + \emptyset_{1.11}Z_{1t-1} + \emptyset_{1.12}Z_{2t-1} + \cdots + \emptyset_{1.1n}Z_{nt-1} + \emptyset_{2.11}Z_{1t-2} + \emptyset_{2.12}Z_{2t-2} + \cdots + \emptyset_{2.1n}Z_{nt-2} + \cdots + \emptyset_{p.11}Z_{1t-p} + \emptyset_{p.12}Z_{2t-p} + \cdots + \emptyset_{p.1n}Z_{nt-p} + \varepsilon_{1t} \quad (3.4)$$

$$\vdots \qquad \vdots \qquad \vdots$$

Case 2:

$if\ i = m; j = n; k = 1, \dots, p$

$$Z_{mt} = \delta_m + \emptyset_{1.m1}Z_{1t-1} + \emptyset_{1.m2}Z_{2t-1} + \emptyset_{1.m3}Z_{3t-1} + \cdots + \emptyset_{1.mn}Z_{nt-1} + \emptyset_{1.m2}Z_{1t-1} + \emptyset_{2.m2}Z_{1t-2} + \emptyset_{2.m3}Z_{3t-2} + \cdots + \emptyset_{2.mn}Z_{nt-2} + \cdots + \emptyset_{p.m1}Z_{1t-p} + \emptyset_{p.m2}Z_{2t-p} + \emptyset_{p.m3}Z_{3t-p} + \cdots + \emptyset_{p.mn}Z_{nt-p} + \varepsilon_{mt} \quad (3.5)$$

Equations (3.4) and (3.5) are the sets of Vector Autoregressive Models defined below,

Mean of Z_{it}

From equation (3.3),

$$Z_{it}(1 - B^k \sum_{k=1}^{p}\sum_{j=1}^{n} \emptyset_{k.ij}) = \delta_i + w_{it}$$

$where\ k = 1, 2, 3, \dots, p\ ; i = 1, 2, 3, \dots, n\ and\ j = 1, 2, 3, \dots, n\ respectively$

$$Z_{it} = \left\{ 1/\left(1 - \sum_{k=1}^{p}\sum_{j=1}^{n} \emptyset_{k.ij} \right) B^k \right\} (\delta_i + w_{it}) \quad (3.6)$$

Where B is the backward shift operator, $\emptyset_{k.ij}$ is a set of model parameters, ε_{it} represents error components.

$$E(Z_{it}) = \quad \delta_i / \left(1 - \sum_{k=1}^{p} \sum_{j=1}^{n} \emptyset_{k.ij} \right) \tag{3.7}$$

$$\text{If } \delta_i = 1, then\ E(Z_{it}) = \ 1/ \left(1 - \sum_{k=1}^{p} \sum_{j=1}^{n} \emptyset_{k.ij} \right) \tag{3.8}$$

expectation of the error term is zero. Therefore, If $\delta_i = 0, then\ E(Z_{it}) = 0$

Variance of Z_{it}

Let Z_{it} in Equation (3.3) be a stationary process that is distributed about the origin such that $E(Z_{it}) = 0, => \delta_1 = \delta_2 = \cdots = \delta_m = 0$. Let the Variance of Z_{it} be defined as $E(Z_{it}Z_{it}) = E(Z_i^2)$, Hence, for $i = 1,2, \dots n$ we have

Case 1: for $i = 1$

$$E(Z_1^2) = \ E\{[\emptyset_{1.11}Z_{1t-1} + \emptyset_{1.12}Z_{2t-1} + \cdots + \emptyset_{1.1n}Z_{nt-1} + \emptyset_{2.11}Z_{1t-2} + \emptyset_{2.12}Z_{2t-2} + \cdots + \emptyset_{2.1n}Z_{nt-2} + \cdots + \emptyset_{p.11}Z_{1t-p} + \emptyset_{p.12}Z_{2t-p} + \cdots + \emptyset_{p.1n}Z_{nt-p} + \varepsilon_{1t}] * [\emptyset_{1.11}Z_{1t-1} + \emptyset_{1.12}Z_{2t-1} + \cdots + \emptyset_{1.1n}Z_{nt-1} + \emptyset_{2.11}Z_{1t-2} + \emptyset_{2.12}Z_{2t-2} + \cdots + \emptyset_{2.1n}Z_{nt-2} + \cdots + \emptyset_{p.11}Z_{1t-p} + \emptyset_{p.12}Z_{2t-p} + \cdots + \emptyset_{p.1n}Z_{nt-p} + \varepsilon_{1t}]\} \tag{3.9}$$

Where the cross product with $\varepsilon_{1t} = 0$

$$\xi_{1t,1t} => \sigma_{1t}^2 \ = \sigma_{1t}^2 \sum_{k=1}^{p} \emptyset_{k.11}^2 + \sum_{k=1}^{p} \sum_{j=2(j\neq1)}^{n} \emptyset_{k.1j}^2 \sigma_{1jt}^2 + 2 \sum_{k=1}^{p} \sum_{j=1}^{n} \sum_{s=1}^{v} \emptyset_{k.1j}\, \emptyset_{k.1s}\, \gamma_{1t,st} + \sigma_{e1t}^2 \tag{3.10}$$

$$\sigma_{1t}^2 \left(1 - \sum_{k=1}^{p} \emptyset_{k.11}^2 \right) = \sum_{k=1}^{p} \sum_{j=2(j\neq1)}^{n} \emptyset_{k.1j}^2 \sigma_{1jt}^2 + 2 \sum_{k=1}^{p} \sum_{j=1}^{n} \sum_{s=1}^{v} \emptyset_{k.1j}\, \emptyset_{k.1s}\, \gamma_{1t,st} + \sigma_{e1t}^2 \tag{3.11}$$

$$\sigma_{1t}^2 = \left(\sum_{k=1}^{p} \sum_{j=2(j\neq1)}^{n} \emptyset_{k.1j}^2 \sigma_{1jt}^2 + 2 \sum_{k=1}^{p} \sum_{j=1}^{n} \sum_{s=1}^{v} \emptyset_{k.1j}\, \emptyset_{k.1s}\, \gamma_{1t,st} + \sigma_{e1t}^2 \right) / \left(1 - \sum_{k=1}^{p} \emptyset_{k.11}^2 \right) \tag{3.12}$$

Where $(s \ \neq 1)$

$\vdots \qquad \vdots \qquad \vdots$

Case 2: $for\ i = m$, it follows that

$\xi_{mt,mt} =>$

$$\sigma_{mt}^2 \ = \sigma_{mt}^2 \sum_{k=1}^{p} \emptyset_{k.mj}^2 + \sum_{k=1}^{p} \sum_{j=1}^{n} \emptyset_{k.mj}^2 \sigma_{mjt}^2 + 2 \sum_{k=1}^{p} \sum_{j=1}^{n} \sum_{s=1}^{v} \emptyset_{k.mj} \emptyset_{k.ms}\, \gamma_{mt,st} + \ \sigma_{emt}^2 \tag{3.13}$$

$$\sigma_{mt}^2 (1 - \sum_{k=1}^{p} \emptyset_{k.mj}^2) \ = \ \sum_{k=1}^{p} \sum_{j=1}^{n} \emptyset_{k.mj}^2 \sigma_{mjt}^2 + 2 \sum_{k=1}^{p} \sum_{j=1}^{n} \sum_{s=1}^{v} \emptyset_{k.mj}\, \emptyset_{k.ms}\, \gamma_{mt,st} + \ \sigma_{emt}^2 \tag{3.14}$$

$$\sigma_{mt}^2 = \left(\sum_{k=1}^{p} \sum_{j=1}^{n} \emptyset_{k.mj}^2 \sigma_{mjt}^2 + 2 \sum_{k=1}^{p} \sum_{j=1}^{n} \sum_{s=1}^{v} \emptyset_{k.mj}\, \emptyset_{k.ms}\, \gamma_{mt,st}, + \sigma_{emt}^2 \right) / \left(1 - \sum_{k=1}^{p} \emptyset_{k.mn}^2 \right)$$

Where $(s \ \neq m)$ (3.15)

Multivariate Autoregressive Distributed Lag (MARDL)

Model Definition

Let $\underline{Z_t} = (Z_{1t}, Z_{2t}, \dots, Z_{mt})'$ be the vector of response time variables, $\underline{\emptyset} = (\emptyset_{k.ij})$ is the vector of coefficients, $\underline{Z_{st}} = (Z_{1t}, Z_{2t}, \dots, Z_{mt})'$ is the vector predictor time variables with their respective coefficients $(\emptyset_{is})$, $\underline{Z_{t-k}} = (Z_{1t-k}, Z_{2t-k}, \dots, Z_{nt-k})'$ be defined as the vector of the predictive lag time variables, $\underline{\delta} = (\delta_1, \delta_2, \dots, \delta_m)'$ is the vector of constants and $\underline{\varepsilon_t} = (\varepsilon_{t1}, \varepsilon_{2t}, \dots, \varepsilon_{mt})'$ is the vector of error terms associated with the vector of response time variables. The Multivariate Autoregressive Distributed Lag Model is presented in the form,

The above definition is presented in the following form,

$\underline{Z_t} = \underline{\delta} + \emptyset_{is}\underline{Z_{st}} \ + \ \varphi_{k.ij}\underline{Z_{jt-k}} + \underline{\varepsilon_t}$ with the expanded form given as,

$$(Z_{1t}, Z_{2t}, \dots, Z_{mt})^I \ = \ (\delta_1, \delta_2, \dots, \delta_m)^I + (\emptyset_{is})(Z_{1t}, Z_{2t}, \dots, Z_{nt})^I + \ (\emptyset_{k.ij})(Z_{1t-k}, Z_{2t-k}, \dots, Z_{nt-k})^I + (\varepsilon_{t1}, \varepsilon_{2t}, \dots, \varepsilon_{mt})^I \tag{3.16}$$

Expansion of Equation (3.16) gives

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \\ \vdots \\ Z_{mt} \end{pmatrix} = \begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} + \begin{pmatrix} 0 & \emptyset_{12} & \cdots & \emptyset_{1n} \\ \emptyset_{21} & 0 & \cdots & \emptyset_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \emptyset_{m1} & \emptyset_{m2} & \cdots & 0 \end{pmatrix} \begin{pmatrix} Z_{1t} \\ Z_{2t} \\ \vdots \\ Z_{mt} \end{pmatrix} + \begin{pmatrix} \emptyset_{1.11} & \emptyset_{1.12} & \cdots & \emptyset_{1.1n} \\ \emptyset_{1.21} & \emptyset_{1.22} & \cdots & \emptyset_{1.2n} \\ \vdots & \vdots & \vdots & \vdots \\ \emptyset_{1.m1} & \emptyset_{1.m2} & \cdots & \emptyset_{1.mn} \end{pmatrix} \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \\ \vdots \\ Z_{mt-1} \end{pmatrix}$$
$$+ \begin{pmatrix} \emptyset_{2.11} & \emptyset_{2.12} & \cdots & \emptyset_{2.1n} \\ \emptyset_{2.21} & \emptyset_{2.22} & \cdots & \emptyset_{2.2n} \\ \vdots & \vdots & \vdots & \vdots \\ \emptyset_{2.m1} & \emptyset_{2.m2} & \cdots & \emptyset_{2.mn} \end{pmatrix} \begin{pmatrix} Z_{1t-2} \\ Z_{2t-2} \\ \vdots \\ Z_{mt-2} \end{pmatrix} + \ldots + \begin{pmatrix} \emptyset_{p.11} & \emptyset_{p.12} & \cdots & \emptyset_{p.1n} \\ \emptyset_{p.21} & \emptyset_{p.22} & \cdots & \emptyset_{p.2n} \\ \vdots & \vdots & \vdots & \vdots \\ \emptyset_{p.m1} & \emptyset_{p.m2} & \cdots & \emptyset_{p.mn} \end{pmatrix} \begin{pmatrix} Z_{1t-p} \\ Z_{2t-p} \\ \vdots \\ Z_{mt-p} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \vdots \\ \varepsilon_{mt} \end{pmatrix} \quad (3.17)$$

More simplified form of Equation (3.17) is,

$$Z_{it} = \delta_i + \sum_{s=1}^{n} \emptyset_{is} Z_{st} + \sum_{k=1}^{p}\sum_{j=1}^{n} \emptyset_{k.ij}\, Z_{jt-k} + \varepsilon_{jt}, i = 1, \ldots, m, (i \neq s) \quad (3.18)$$

Where $\emptyset_{is}$ is a coefficient matrix of the non-lag predictor variables, $\emptyset_{k.ij}$ are matrices of coefficients of j predictors to i responses at k lags, $\delta_{i(i=1,\ldots,m)}$ are constants.

Model Expansion: Give Equation (3.18)

$$Z_{it} = \delta_i + \sum_{s=1}^{n} \emptyset_{is} Z_{st} + \sum_{k=1}^{p}\sum_{j=1}^{n} \emptyset_{k.ij}\, Z_{jt-k} + \varepsilon_{jt}, i = 1, \ldots, m\ (i \neq s)$$

Case 1:

$if\ i = 1; s = 1, \ldots, n; j = 1, \ldots, n; k = 1, \ldots, p$

$$Z_{1t} = \delta_1 + \emptyset_{12} Z_{2t} + \emptyset_{13} Z_{3t} + \ldots + \emptyset_{1n} Z_{nt} + \emptyset_{1.11} Z_{1t-1} + \emptyset_{1.12} Z_{2t-1} + \cdots + \emptyset_{1.1n} Y_{nt-1} + \emptyset_{2.11} Z_{1t-2} + \emptyset_{2.m2} Z_{2t-2} + \cdots + \emptyset_{2.1n} Z_{nt-2} + \cdots + \emptyset_{p.11} Z_{1t-p} + \emptyset_{p.12} \emptyset_{2t-p} + \cdots + \emptyset_{p.1n} Z_{nt-p} + \varepsilon_{1t} \quad (3.19)$$

Case 2:

$if\ i = m; s = 1, \ldots, n-1 (s \neq m);\ j = n; k = 1, \ldots, p$ we have,

$$Z_{mt} = \delta_m + \emptyset_{m1} Z_{1t} + \emptyset_{m2} Z_{2t} + \ldots + \emptyset_{m(n-1)} Z_{(n-1)t} + \emptyset_{1.m1} Z_{1t-1} + \emptyset_{1.m2} Z_{2t-1} + \cdots + \emptyset_{1.mn} Y_{nt-1} + \emptyset_{2.m1} Z_{1t-2} + \emptyset_{2.m2} Z_{2t-2} + \cdots + \emptyset_{2.mn} Z_{nt-2} + \cdots + \emptyset_{p.m1} Z_{1t-p} + \emptyset_{p.m2} \emptyset_{2t-p} + \cdots + \emptyset_{p.mn} Z_{nt-p} + \varepsilon_{mt} \quad (3.20)$$

Equations (3.19) and (3.20) are a set of Multivariate Autoregressive Distributed Lag (MARDL) Models

Mean of Z_{it}

Let Equation (3.18) be presented as the combination of the two components,

$$\sum_{s=1}^{n} \emptyset_{is} Z_{st} + v_{it} = E \text{ and } \delta_i + \sum_{k=1}^{p}\sum_{j=1}^{n} \emptyset_{k.ij}\, Z_{jt-k} + w_{it} = F;\ Where\ Z_{it} = E + F$$

E and F represent Multivariate Linear Regression (MLR) and VAR models respectively.

$But\ E(Z_{it}) = E(E) + E(F)$

E(E) is given as;

For $i = 1, s = 2, 3, \ldots, n\ (i \neq s)$

$E(Z_{1t}) = \emptyset_{12} E(Z_{2t}) + \emptyset_{13} E(Z_{3t}) + \cdots + \emptyset_{1n} E(Z_{nt}) + E(v_{1t})$

$E(Z_{1t}) = \emptyset_{12}\mu_2 + \emptyset_{13}\mu_3 + \cdots + \emptyset_{1n}\mu_n + E(v_{1t})$

But $E(v_{1t}) = 0$

$E(Z_{1t}) = \emptyset_{12}\mu_2 + \emptyset_{13}\mu_3 + \cdots + \emptyset_{1n}\mu_n$

$$E(Z_{1t}) = \sum_{s=2}^{n} \emptyset_{is}\mu_s, \text{ where } \mu_s \geq 1 \quad (3.21)$$

$$\vdots \qquad \vdots \qquad \vdots$$

For $i = m, s = 1, 3, \ldots, n-1$.

$E(Z_{nt}) = \emptyset_{n1} E(Z_{1t}) + \emptyset_{n2} E(Z_{2t}) + \cdots + \emptyset_{n(n-1)} E(Z_{(n-1)t}) + E(v_{nt})$

$E(Z_{nt}) = \emptyset_{n1}\mu_1 + \emptyset_{n2}\mu_2 + \cdots + \emptyset_{n(n-1)}\mu_{n-1} + E(v_{nt})$

But $E(v_{mt}) = 0$

$E(Z_{nt}) = \emptyset_{n1}\mu_1 + \emptyset_{n2}\mu_2 + \cdots + \emptyset_{n(n-1)}\mu_{n-1}$

$$E(Z_{nt}) = \sum_{s=1}^{n-1} \emptyset_{is}\mu_s, \text{ where } \mu_s \geq 1 \text{ and } (i \neq s) \quad (3.22)$$

Combining Equations (3.7) and (3.22) produces

$$E(Z_{it}) = \sum_{s=1}^{n} \emptyset_{is}\mu_s + \left[\delta_i / \left(1 - \sum_{k=1}^{p}\sum_{j=1}^{n} \emptyset_{k.ij}\right)\right], \qquad \delta_i,\ \mu_s \geq 1\ ;\ i \neq s \qquad (3.23)$$

Which is the mean of the Multivariate Autoregressive Distributed Lag Model.

Variance of Multivariate Autoregressive Distributed Lag (MARDL) Model

Let Z_{it} in Equation (3.18) be a stationary process that is distributed about the origin such that $E(Z_{it}) = 0, => \delta_1 = \delta_2 = \cdots = \delta_m = 0$. Let the Variance of Z_{it} be defined as $E(Z_{it}Z_{it}) = E(Z_i^2)$, Hence, for $i = 1,2, \ldots n$ we have

Case 1: for $i = 1;\ s = 1, \ldots, n; j = 1, \ldots, n; k = 1, \ldots, p$

$$E(Z_1^2) = E\{[\emptyset_{12}Z_{2t} + \emptyset_{13}Z_{3t} + \ldots + \emptyset_{1n}Z_{nt} + \emptyset_{1.11}Z_{1t-1} + \emptyset_{1.12}Z_{2t-1} + \cdots + \emptyset_{1.1n}Y_{nt-1} + \emptyset_{2.11}Z_{1t-2} + \emptyset_{2.12}Z_{2t-2} + \cdots + \emptyset_{2.1n}Z_{nt-2} + \cdots + \emptyset_{p.11}Z_{1t-p} + \emptyset_{p.12}\emptyset_{2t-p} + \cdots + \emptyset_{p.1n}Z_{nt-p} + \varepsilon_{1t}] * [\emptyset_{12}Z_{2t} + \emptyset_{13}Z_{3t} + \ldots + \emptyset_{1n}Z_{nt} + \emptyset_{1.11}Z_{1t-1} + \emptyset_{1.12}Z_{2t-1} + \cdots + \emptyset_{1.1n}Y_{nt-1} + \emptyset_{2.11}Z_{1t-2} + \emptyset_{2.12}Z_{2t-2} + \cdots + \emptyset_{2.1n}Z_{nt-2} + \cdots + \emptyset_{p.11}Z_{1t-p} + \emptyset_{p.12}\emptyset_{2t-p} + \cdots + \emptyset_{p.1n}Z_{nt-p} + \varepsilon_{1t}]\} \qquad (3.24)$$

$$\xi_{1t,1t} => \sigma_{1t}^2 + \left[\sigma_{1jt}^2\left(\sum_{j=2}^{n}\emptyset_{1j}^2 + \sum_{k=1}^{p}\sum_{j=2}^{n}\emptyset_{k.1j}^2\right) + 2\sum_{r=2}^{n}\sum_{s=r+1}^{m}\emptyset_{1r}\emptyset_{1s}\gamma_{1s} + 2\sum_{k=1}^{p}\sum_{r\neq 2}^{n}\sum_{j=r+1}^{v}\emptyset_{1r}\emptyset_{k.1j}\gamma_{r,j(k)} + 2\sum_{k=1}^{p}\sum_{l=1}^{q}\sum_{j\neq 1}^{m}\sum_{v=j+1}^{u}\emptyset_{k.1j}\emptyset_{k.1v}\gamma_{r(l),j(k)} + \sigma_{e1t}^2\right] / \left(1 - \sum_{k=1}^{p}\emptyset_{k.11}^2\right)$$

$$where\ m = n = u \quad (3.25)$$

$$\vdots \qquad \vdots \qquad \vdots$$

Case 2:

$for\ i = m,$ it follows that

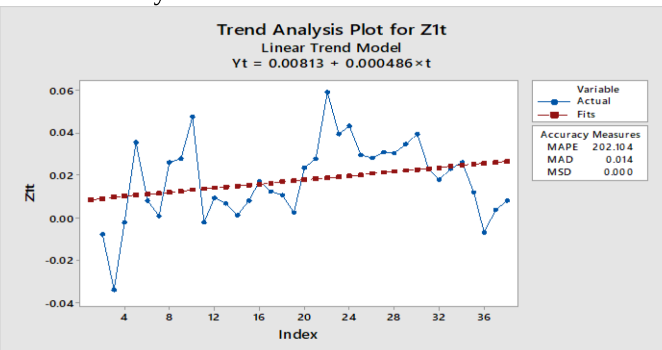
$$\xi_{mt,mt} = \sigma_{mt}^2 = \left[\sigma_{mjt}^2\left(\sum_{j=1(j\neq m)}^{n}\emptyset_{mj}^2 + \sum_{k=1}^{p}\sum_{j=1}^{n}\emptyset_{k.mn}^2\right) + 2\sum_{r=1}^{n}\sum_{s=r+1}^{m}\emptyset_{mr}\emptyset_{ms}\gamma_{ms} + 2\sum_{k=1}^{p}\sum_{r\neq 1}^{n}\sum_{j=r+1}^{m}\emptyset_{mr}\emptyset_{k.mj}\gamma_{r,j(k)} + 2\sum_{k=1}^{p}\sum_{l=1}^{q}\sum_{j\neq 1}^{m}\sum_{v=j+1}^{u}\emptyset_{k.mj}\emptyset_{k.mv}\gamma_{r(l),j(k)} + \sigma_{emt}^2\right] / \left(1 - \sum_{k=1}^{p}\emptyset_{k.mn}^2\right) \qquad where\ m = n = u\ (3.26)$$

Equations (3.25) and (3.26) are the variances of the Multivariate Autoregressive Distributed Lag (MARDL) Model.

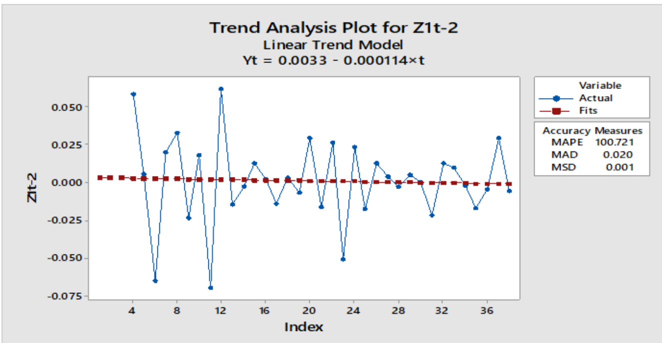
Data and Methodology

Data from CBN Statistical Bulletin covering the period from 1988 to 2020 on some Nigeria's microeconomic variables like agriculture (Z_{1t}), petroleum / crude oil (Z_{2t}), telecommunication (Z_{3t}) and gross domestic products (Z_{4t}) was used for the analysis and estimation of model parameters.

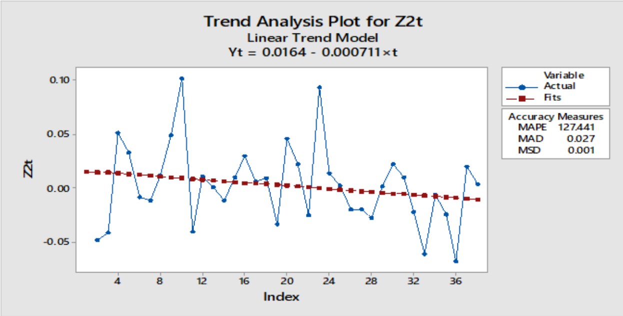
The trend analysis of these microeconomic was shown below



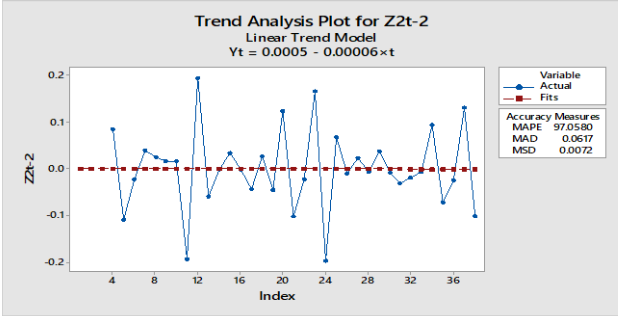
Trend Analysis of Z_{1t} before differencing.



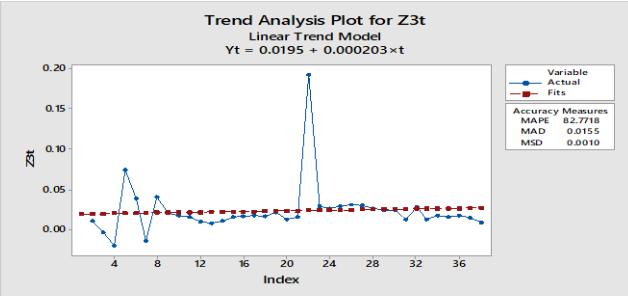
Trend Analysis of Z_{1t} after differencing.



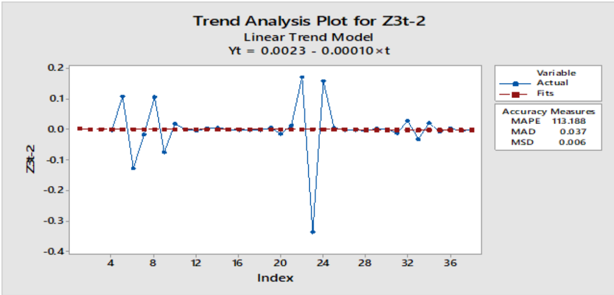
Trend analysis of Z_{2t} before differencing.



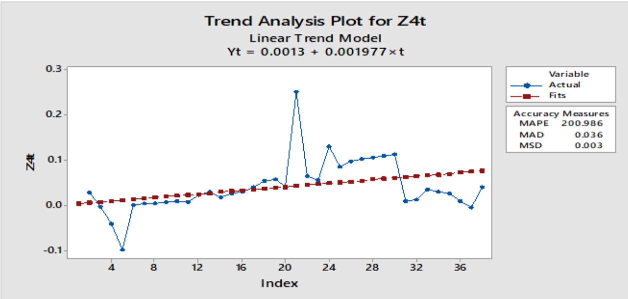
Trend Analysis of Z_{2t} after differencing.



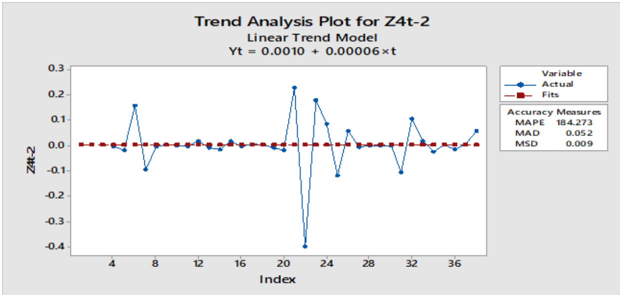
Trend analysis of Z_{3t} before differencing.



Trend analysis of Z_{3t} after differencing.



Trend analysis of Z_{4t} before differencing.



Trend analysis of Z_{4t} after differencing.

The ACF and PACF plots indicated a cutoff at lag order p=2. Furthermore, the model selection criteria AIC, BIC, and SIC evaluated for both models identified the MARDL model as the most appropriate. The model estimation was carried out using Minitab statistical software.

Empirical Results

In this section, we examine outcomes derived from the two models VAR and MARDL. These encompass the mean and variances from both models. The findings are showcased in the tables provided below

S/N		VAR	MARDL
1	δ_1	0.01949	0.00118
2	δ_2	0.00327	0.00870
3	δ_3	0.02425	0.00552
4	δ_4	0.04229	0.0166
5	$E(Z_{1t})$	0.01891	0.02069
6	$E(Z_{2t})$	0.00505	0.01162
7	$E(Z_{3t})$	0.001819	0.0291
8	$E(Z_{4t})$	0.029993	0.10276
9	$\sigma^2_{\varepsilon 1t}$	0.0000000000000000114	0.000000000000011
10	$\sigma^2_{\varepsilon 2t}$	0.000000000000000016	0.000000000000028
11	$\sigma^2_{\varepsilon 3t}$	0.000000000000000013	0.000000000000011
12	$\sigma^2_{\varepsilon 4t}$	0.000000000000000011	0.000000000000028
13	$\xi_{1t,1t}$	0.0503	0.000262
14	$\xi_{2t,2t}$	0.0112	0.0019012
15	$\xi_{3t,3t}$	0.0122	0.0019145
16	$\xi_{4t,4t}$	0.0007	0.0001517

Discussion and Conclusion

This research provides further insights into the properties of Multivariate Autoregressive Distributed Lag (MARDL) models, contributing to their theoretical development and practical application. Specifically, it examines and compares the properties of MARDL and VAR models to assess their relative performance. The comparison of the mean properties of both models, indicates no significant difference between them. However, an analysis of their variance properties reveals that the MARDL model exhibits smaller variances than the VAR model. This suggests that the MARDL model is more stable, controlled, and consistent, making it more reliable for prediction and forecasting.

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